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On Variational Aspects of Some Nonconvex Nonsmooth Global Optimization Problem

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Abstract. The paper is devoted to study a global optimization problem for a class of nonsmooth, nonconvex and non–locally Lipschitz functionals on a vector–valued reflexive Banach space. The existence of an element which is both a global minimizer and a solution of the associated hemivariational inequality is proved under some unilateral growth restrictions imposed on "nonlinearities" and under the assumption that an appropriately modified version of the Hedberg truncation procedure can be applied.

Key words: Global Optimization Problem, Hemivariational Inequality

1 Introduction

The purpose of this paper is to study a class of nonsmooth, nonconvex and non–locally Liposchitz functionals on a reflexive Banach space V. We are interested in establishing the existence of global minimizers and in providing a detailed exposition of their variational properties. It is well known that if $I : V \to R$ is smooth then the necessary condition for the optimality is the Euler–Lagrange equation $I'(u) = 0$, $I'(u)$ being a derivative of I at u. If I is assumed to be convex then its minimizer is characterized by the condition $0 \in \partial I(u)$, where $\partial I(u) \subset V^*$ is a subgradient of I at u. This condition gives rise to variational inequalities, well known in the literature (see for instance [8, 9, 19, 26]). In more general cases we are led to the condition $0 \in \partial I(u)$, where $\partial I(u)$ is understood as to be the generalized gradient of Clarke–Rockafellar [7, 27] The associated variational expressions are then called hemivariational inequalities. They have been first introduced and studied by Panagiotopoulos in [19–24]. For the further results in this area we refer the reader to [1, 2, 6, 11–18, 25, 28, 29].

The present approach is devoted to study a global optimizaton problem and to examine the associated hemivariational inequality for a class of nonsmooth, nonconvex and non–locally Lipschitz functionals on a vector–valued reflexive Banach space. The paper uses some ideas from Webb [29] and Brezis and Browder [4] who studied strongly nonlinear equations in scalar–valued Sobolev spaces by applying Hedberg's approximation technique [10]. It is of our interest to consider in this paper the case in which functionals involved are nonsmooth and defined on vector–valued function spaces.

To precise the class of functionals we are going to deal with let us suppose that V is a reflexive Banach space compactly imbedded into $L^p(\Omega; R^N)$, $p > 1$, $N \geq 1$,

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Ω is a bounded domain in R^m, $m \geq 1$. We write $||\cdot||_V$ and $||\cdot||_{L^p(\Omega;R^N)}$ for the norms in V and $L^p(\Omega; R^N)$, respectively. Throughout the paper it is assumed that $V \cap L^\infty(\Omega; R^N)$ is dense in V.

Let $\Phi : V \to R$ be a smooth function from V into R. By $\Phi' : V \to V^\star$ we denote its derivative. Here $V^\star$ stands for the dual of V. For the pairing over $V^\star \times V$ we use the symbol $\langle\cdot,\cdot\rangle$. Throughout the paper it will be assumed that Φ' is coercive and bounded. It means that

$$\langle\Phi'(u), u\rangle \geq a(||u||_V)||u||_V \quad \forall u \in V$$

for some $a : R^+ \to R$, $R^+ = \{x \in R : x \geq 0\}$ with $\lim_{r\to\infty} a(r) = +\infty$, and that Φ' mapps bounded sets of V into bounded sets of $V^\star$. Moreover, we assume that Φ' is pseudo–monotone. It implies that whenever $u_n \to u$ weakly in V and $\limsup\langle\Phi'(u_n), u_n - u\rangle \leq 0$ then $\Phi'(u_n) \to \Phi'(u)$ weakly in $V^\star$ and $\langle\Phi'(u_n), u_n\rangle \to \langle\Phi'(u), u\rangle$. Further, let $j : R^N \to R$ be a locally Lipschitz function from R^N into R. Throughtout the paper we assume j to satisfy two unilateral growth restrictions:

(H1)
$$j^0(\xi; -\xi) \leq \beta(1 + |\xi|) \quad \forall\xi \in R^N;$$

(H2)
$$j^0(\xi; \eta - \xi) \leq \alpha(r)(1 + |\xi|^\sigma) \quad \forall\xi, \eta \in R^N,\ |\eta| \leq r,\ r > 0,$$

where $\alpha : R^+ \to R^+$ is a nondecreasing function from R^+ into R^+, $j^0(\cdot,\cdot)$ is the generalized Clarke differential [7], i.e.

$$j^0(\xi;\eta) = \limsup_{\substack{h\to 0\\ \lambda\to 0}} \frac{j(\xi + h + \lambda\eta) - j(\xi + h)}{\lambda},$$

$1 \leq \sigma < p$ and where $\beta \geq 0$ is a nonnegative constant. By means of j we can define a functional $J : V \to \bar{R} = R \cup \{+\infty\}$ by

$$J(u) = \begin{cases} \int_\Omega j(u)d\Omega & \text{if } j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Now we are in a position to precise the class of functionals we are going to deal with. Let $g \in V^\star$ be an arbitrary element of $V^\star$. Define $I : V \to \bar{R}$ by the formula

$$I(u) = \Phi(u) + \int_\Omega j(u)d\Omega \ - \langle g, u\rangle, \ \ u \in V. \tag{1.2}$$

One can check immediately that under the hypotheses $(H1)$ and $(H2)$, J is a nonconvex, nonsmooth, not locally Lipschitz functional with the effective domain $D(I) = \{v \in V : J(u) < \infty\}$ not coinciding with the whole space V. So I shares all the aforementioned irregularities.

The following minimization problem will be considered.

Problem (P) Find $u \in V$ such that

$$I(u) = \inf_{v \in V} I(v). \tag{1.3}$$

It is natural to associate with (1.3) the following hemivariational inequality.

$$\langle \Phi'(u) - g, v - u \rangle + \int_\Omega j^0(u; v - u) d\Omega \geq 0 \quad \forall v \in V. \tag{1.4}$$

Since growth restrictions imposed on j as formulated in $(H1)$ and $(H2)$ are too weak to ensure *a priori* the finite integrability of $j^0(u; v-u)$ in Ω for any $v \in V$, we have to modify (1.4) and to consider the following problem.

Problem (Q) Find $u \in V$ for which there exists $\chi \in L^1(\Omega; R^N)$ with the properties that

$$\langle \Phi'(u) - g, v - u \rangle + \int_\Omega \chi \cdot (v - u) d\Omega = 0 \quad \forall v \in V \cap L^\infty(\Omega; R^N) \tag{1.5}$$

$$\chi \cdot u \in L^1(\Omega), \quad \chi \in \partial j(u), \quad j(u) \in L^1(\Omega). \tag{1.6}$$

Here and from now on " $\cdot$ " denotes the inner product in R^N. Our main task in this paper is to show that if the original space V possesses some approximation property, then there exists at least one element $u \in V$ which is a solution of both (P) and (Q). The mentioned hypothesis is formulated as follows.

Truncation Hypothesis For each $u \in V$ there exist two sequences $\{\varepsilon_k\} \subset L^\infty(\Omega)$ and $\{\kappa_k\} \subset L^\infty(\Omega; R^N)$ with $0 \leq \varepsilon_k \leq 1$ and $||\kappa_k||_{L^\infty(\Omega; R^N)} \leq C$ such that

$$\textbf{(H3)} \qquad \begin{gathered} \{(1-\varepsilon_k)u + \varepsilon_k \kappa_k\} \subset V \cap L^\infty(\Omega; R^N) \\ (1-\varepsilon_k)u + \varepsilon_k \kappa_k \to u \ \ in \ \ V. \end{gathered}$$

One can easily recognize that in the scalar case $N = 1$ the property described in $(H3)$ is common for Sobolev spaces $W^{m,p}(\Omega)$ provided the boundary $\partial\Omega$ is smooth enough. This is a consequence of the famous result of Hedberg [10] who showed that each element $u \in W^{m,p}(\Omega)$ can be approximated by functions of the form $(1 - \varepsilon_k)u \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$ with $0 \leq \varepsilon_k \leq 1$ in the sense that $(1 - \varepsilon_k)u \to u$ in $W^{m,p}(\Omega)$.

2 Some Preliminaries and Global Optimization Problem

This Section is devoted to study the global optimization problem (P). First we point out some properties of j resulting from $(H1)$ and $(H2)$.

Lemma 2.1 Assume that

$$j^0(\xi;-\xi) \le \beta(1+|\xi|^s) \quad \xi \in R^N, \quad s \ge 1. \tag{2.1}$$

Then there exists a $k \ge 0$ such that

$$j(\xi) \ge -k(1+|\xi|^s) \quad \xi \in R^N. \tag{2.2}$$

Proof. For any $\xi \in R^N$ with $|\xi| > 1$ define $\phi(t) = j((1-t)\xi)$. Then by the Lebourg mean value theorem one can get

$$j(\frac{\xi}{|\xi|}) - j(\xi) = \phi(\frac{|\xi|-1}{|\xi|}) - \phi(0) \le \phi^0(\theta; \frac{|\xi|-1}{|\xi|}) \le j^0((1-\theta)\xi; -\xi)\frac{|\xi|-1}{|\xi|}$$

for some $0 < \theta < \frac{|\xi|-1}{|\xi|}$, where $\phi^0(\cdot,\cdot)$ stands for the generalized Clarke differential [7]. Hence

$$\begin{aligned} j(\frac{\xi}{|\xi|}) - j(\xi) &\le j^0((1-\theta)\xi; -(1-\theta)\xi)\frac{|\xi|-1}{(1-\theta)|\xi|} \\ &\le \beta(1+(1-\theta)^s|\xi|^s)\frac{|\xi|-1}{(1-\theta)|\xi|} \\ &\le \frac{\beta}{1-\theta} + \beta(1-\theta)^{s-1}|\xi|^s \le \beta|\xi| + \beta|\xi|^s \\ &\le 2\beta(1+|\xi|^s), \end{aligned}$$

because $\frac{1}{1-\theta} \le |\xi|$. Thus , for an arbitrary $\xi \in R^N$ the estimate follows

$$\begin{aligned} j(\xi) \ge j(\frac{\xi}{|\xi|}) - 2\beta(1+|\xi|^s) \ge - \sup_{|\eta|\le 1} |j(\eta)| - 2\beta(1+|\xi|^s) \\ \ge -k(1+|\xi|^s), \end{aligned}$$

where $k = \sup_{|\eta|\le 1} |j(\eta)| + 2\beta$. The proof is complete.

Lemma 2.2 Suppose that $(H2)$ holds. Then there exists a nondecreasing function $\gamma : R^+ \to R^+$ such that the estimate

$$j((1-\varepsilon)\xi + \varepsilon\eta) \le \begin{cases} j(\xi) + \varepsilon\gamma(|\eta|)(1+|\xi|^\sigma) & if \ \ 0 \le \varepsilon \le \frac{|\xi|-1}{|\xi|} \\ \sup_{|\mu|\le|\eta|+1} j(\mu) & if \ \ \frac{|\xi|-1}{|\xi|} < \varepsilon \le 1 \end{cases} \tag{2.3}$$

is valid for any $\xi,\ \eta \in R^N$ with $|\xi| > 1$.

Proof. Let $\xi, \eta \in R^N$ and $\phi(t) = j((1-t)\xi + t\eta)$, $t \in [0,1]$. Take $|\xi| > 1$ and $0 \le \varepsilon \le \frac{|\xi|-1}{|\xi|}$. The mean value theorem of Lebourg allows us to conclude that

$$j((1-\varepsilon)\xi + \varepsilon\eta) - j(\xi) = \phi(\varepsilon) - \phi(0) \le \phi^0(\theta, \varepsilon) \le \varepsilon j^0((1-\theta)\xi + \theta\eta; -\xi + \eta)$$

for some $0 < \theta < \varepsilon \le \frac{|\xi|-1}{|\xi|}$. Hence due to $(H2)$ one obtains

$$\begin{aligned} j((1-\varepsilon)\xi + \varepsilon\eta) - j(\xi) &\le \varepsilon j^0((1-\theta)\xi + \theta\eta; \eta - (1-\theta)\xi - \theta\eta)\frac{1}{1-\theta} \\ &\le \varepsilon\alpha(|\eta|)(1 + |(1-\theta)\xi + \theta\eta|^\sigma)\frac{1}{1-\theta} \\ &\le \varepsilon\alpha(|\eta|)(1 + c(1-\theta)^\sigma|\xi|^\sigma + c\theta^\sigma|\eta|^\sigma)\frac{1}{1-\theta} \\ &\le \varepsilon\alpha(|\eta|)(\frac{1}{1-\theta} + c(1-\theta)^{\sigma-1}|\xi|^\sigma + c|\eta|^\sigma\frac{1}{1-\theta}) \\ &\le \varepsilon\alpha(|\eta|)(|\xi| + c(|\xi|^\sigma + c|\eta|^\sigma|\xi|) \\ &\le \varepsilon\alpha(|\eta|)(c|\eta|^\sigma + c + 1)(1 + |\xi|^\sigma), \end{aligned}$$

where c is a positive constant. Here we have used the fact that the condition $0 \le \theta \le \frac{|\xi|-1}{|\xi|}$ implies $\frac{1}{1-\theta} \ge |\xi|$. If $\frac{|\xi|-1}{|\xi|} < \varepsilon \le 1$ then $|(1-\xi)\varepsilon + \varepsilon\eta| \le 1 + |\eta|$ and the second estimate in (2.3) follows immediately. Finally, setting

$$\gamma(|\eta|) = \alpha(|\eta|)(c|\eta|^\sigma + c + 1)$$

we get (2.3) which completes the proof.

By the local Lipschitz property of j we verify easily that

$$j((1-\varepsilon)\xi + \varepsilon\eta)) \le j(\xi) + \varepsilon L(2 + |\eta|)(2 + |\eta|) \quad \text{for } |\xi| \le 2,\ \ 0 \le \varepsilon \le 1, \tag{2.4}$$

where $L(r)$ denotes the Lipschitz constant of j in the ball $B(0,r) = \{\zeta \in R^N : |\zeta| \le r\}$, $r \ge 0$. This combined with (2.3) permits the formulation of the following estimate

$$j((1-\varepsilon) + \varepsilon\eta) \le j(\xi) + \varepsilon\tilde{\gamma}(|\eta|)(1 + |\xi|^\sigma) \quad \text{for } 0 \le \varepsilon \le \frac{1}{2}, \tag{2.5}$$

where

$$\tilde{\gamma}(|\eta|) := \gamma(|\eta|) + L(2 + |\eta|)(2 + |\eta|).$$

It is worth noting that (2.5) holds for $\varepsilon \in [0, \frac{1}{2}]$ not for the whole interval $[0,1]$. In the sequel we shall need also an estimate similar to (2.5) but with right hand side independent of $\varepsilon \in [0,1]$. It is not difficult to verify that under the hypotheses $(H1)$ and $(H2)$ Lemma 2.2 combined with (2.4) yields

$$j((1-\varepsilon)\xi + \varepsilon\eta) \leq j(\xi) + \varphi(|\eta|)(1+|\xi|^{\sigma}) \quad \text{for } 0 \leq \varepsilon \leq 1, \tag{2.6}$$

where $\varphi(\cdot)$ is defined by

$$\varphi(|\eta|) := \gamma(|\eta|) + L(2+|\eta|)(2+|\eta|) + \sup_{|\mu|\leq|\eta|+1} j(\mu) + k \quad \eta \in R^N.$$

Let us turn to the global optimization problem (P). First we point out some properties of I. The estimate (2.2) combined with Fatou's lemma allows us to conclude that $J : V \to \bar{R}$ is weakly lower semicontinuous. Since Φ has the same property due to the pseudo-monotonicity of Φ' (cf. [5], see also [18]), thus so is the functional I. The coercivity of Φ' and (2.1) imply that I is bounded from below. Indeed, one can check easily that

$$\begin{aligned} I(v) &= \Phi(v) + \int_{\Omega} j(v)d\Omega - \langle g, v\rangle \\ &\geq \Phi(0) + \int_0^1 \langle \Phi'(tv), v\rangle dt - \int_{\Omega} k(1+|v|)d\Omega - ||g||_{V^*}||v||_V \\ &\geq \Phi(0) + \int_0^1 a(t||v||_V)||v||_V dt - k \operatorname{mes}\Omega - k_1||v||_V \\ &\geq \Phi(0) + \tilde{a}(||v||_V)||v||_V - k\operatorname{mes}\Omega - k_1||v||_V, \quad k_1 \geq 0, \end{aligned}$$

where the function

$$\tilde{a}(r) = \frac{1}{r}\int_0^r a(t)dt \quad r \geq 0$$

satisfies the coercivity condition $\tilde{a}(r) \to +\infty$ as $r \to +\infty$. Accordingly, I is coercive as well and by the classical argument the infimum of I over V is achieved.

Theorem 2.1 Suppose that the hypothesis $(H1)$ holds. Then problem (P) has at least one solution.

The result below will prove extremely useful in the next section where we pass into the framework of the $V \cap L^{\infty}(\Omega; R^N)$–space.

Proposition 2.2 Suppose that $(H1) \div (H3)$ hold and that $u \in V$ is such that

$$I(u) = \inf_{v \in V\cap L^{\infty}(\Omega;R^N)} I(v). \tag{2.7}$$

Then u is a solution of (P).

Proof. It is clear that the assertion will be proved if we show that whenever $J(w)$

is finite at $w \in V$, one can find a sequence $\{w_k\} \subset V \cap L^\infty(\Omega; R^N)$ such that $w_k \to w$ in V and

$$J(w_k) \to J(w).$$

Recall that by $(H3)$ there exist $\{\varepsilon_k\} \subset L^\infty(\Omega)$ and $\{\kappa_k\} \subset L^\infty(\Omega; R^N)$ with $0 \le \varepsilon_k \le 1$ and $||\kappa_k||_{L^\infty(\Omega;R^N)} \le C$, such that

$$\{(1-\varepsilon_k)w + \varepsilon_k\kappa_k\} \subset V \cap L^\infty(\Omega; R^N)$$
$$(1-\varepsilon_k)w + \varepsilon_k\kappa_k := w_k \to w \quad \text{in} \;\; V.$$

We can certainly assume that

$$w_k \to w \quad \text{a.e. in } \Omega,$$

for the imbedding $V \subset L^p(\Omega; R^n)$ is compact. From (2.2) and (2.6) we have

$$-k(1+|w_k|) \le j(w_k) \le j(w) + \varphi(C)(1+|w|^\sigma).$$

Since $j(w) \in L^1(\Omega)$ and

$$\lim_k \int_\Omega (1+|w_k|^s)d\Omega = \int_\Omega (1+|w|^s)d\Omega, \quad 1 \le s \le p,$$

it follows by Fatou's lemma that

$$\liminf_k \int_\Omega j(w_k)d\Omega \ge \int_\Omega j(w)d\Omega$$

and

$$\limsup_k \int_\Omega j(w_k)d\Omega \le \int_\Omega j(w)d\Omega.$$

Therefore

$$\lim_k \int_\Omega j(w_k)d\Omega = \int_\Omega j(w)d\Omega,$$

and the proof is complete.

3 Variational Properties

This Section provides a detailed exposition of variational properties of solutions of the global optimization problem (P). First result in this direction reads.

Proposition 3.1 Suppose that $(H1)$ and $(H2)$ hold and that $u \in V$ is a solution of (P). Then the following hemivariational inequality is fulfilled

$$\langle \Phi'(u) - g, v-u\rangle + \int_\Omega j^0(u; v-u)d\Omega \quad \forall v \in V \cap L^\infty(\Omega; R^N). \tag{3.1}$$

Proof. Due to the optimality for each $v \in V \cap L^\infty(\Omega; R^N)$ and $0 < t \leq \frac{1}{2}$ we have

$$I(u) \leq I(u + t(v - u))$$

with finite values on the right hand side of the foregoing inequality. Indeed, from (2.5) it follows that

$$\frac{J(u + t(v-u)) - J(u)}{t} \leq \gamma(||v||_{L^\infty(\Omega;R^N)}) \int_\Omega (1 + |u|^\sigma) d\Omega < +\infty. \tag{3.2}$$

Further, for any $0 < t \leq \frac{1}{2}$ one obtains the inequality

$$0 \leq \frac{\Phi(u + t(v-u)) - \Phi(u)}{t} + \int_\Omega \frac{j(u + t(v-u)) - j(u)}{t} d\Omega - \langle g, v - u \rangle.$$

Before passing to the limit as $t \to 0_+$ we make the observation that due to (2.5) one can apply Fatou's lemma to the integral over Ω . It amounts to

$$\limsup_{t \to 0_+} \int_\Omega \frac{j(u + t(v-u)) - j(u)}{t} d\Omega \leq \int_\Omega j^0(u; v - u) d\Omega < \infty. \tag{3.3}$$

Hence, by application of the limit procedure as $t \longrightarrow 0_+$ the assertion results.

Note that (3.1) is not valid for every $v \in V$. Since there are no any *a priori* arguments ensuring the finite integrability of $j^0(u; v - u)$ in Ω we are allowed to substitute there only elements of $V \cap L^\infty(\Omega; R^N)$.

Now we proceed to study problem (Q). The idea is to pass into the framework of finite dimensional subspaces of $V \cap L^\infty(\Omega; R^N)$. Let Λ be the family of all finite dimensional subspaces F of $V \cap L^\infty(\Omega; R^N)$, ordered by inclusion. Denote by $i_F : V \to F$ the inclusion mapping of F into V, and by $i_F^\star : V^\star \to F^\star$ the dual projection mapping of $V^\star$ into $F^\star$, $F^\star$ being the dual of F. Define $I_F : F \to \bar{R}$ by

$$I_F(v) = I(i_F v) \quad v \in F.$$

For any $F \in \Lambda$ we formulate the finite dimensional minimization problem.

Problem (P_F) Find $u_F \in F$ such that

$$I_F(u_F) = \inf_{v \in F} I_F(v). \tag{3.4}$$

Proposition 3.3 For every $F \in \Lambda$ the problem (P_F) has at least one solution. Moreover, there exists a positive constant C not depending on F such that

$$||u_F|| \leq M. \tag{3.5}$$

Proof. The existence of a solution to (P_F) is obvious. For the boundedness we make the observation that the restriction of J to $L^\infty(\Omega; R^N)$, say J_∞, is locally Lipschitz. Moreover,

$$I_F = \Phi \circ i_F - \langle i_F^\star g, \cdot \rangle_F + J_\infty \circ i_F,$$

where $\langle \cdot, \cdot \rangle_F$ denotes the pairing over $F^\star \times F$, is a locally Lipschitz function on F, as well. Thus, if u_F is a solution of (P_F) then

$$0 \in \partial I_F(u_F),$$

where $\partial I_F(u_F)$ stands for the generalized Clarke gradient of I_F at u_F [7]. By the basic calculus it follows directly that

$$0 \in \partial I_F(u_F) \subset i_F^\star \Phi'(u_F) - i_F^\star g + i_F^\star \partial J_\infty(u_F).$$

Since $\partial J_\infty(u_F) \subset L^1(\Omega; R^N)$, [7], there exists $\chi_F \in L^1(\Omega; R^N)$ such that

$$\begin{gathered} \langle \Phi'(u_F) - g, v - u_F \rangle + \int_\Omega \chi_F \cdot (v - u_F) d\Omega = 0 \quad \forall v \in F \\ \chi_F \in \partial j(u_F). \end{gathered} \tag{3.6}$$

Substituting $v = 0$ into (3.6) and applying $(H1)$ yields

$$\begin{aligned} \langle \Phi'(u_F), u_F \rangle \le \langle g, u_F \rangle + \int_\Omega \chi_F \cdot (-u_F) d\Omega &\le ||g||_{V^*} ||u_F||_V + k \int_\Omega (1 + |u_F|) d\Omega \\ &\le ||g||_{V^*} ||u_F||_V + k \,\mathrm{mes}\,(\Omega) + k_1 ||u_F||_V. \end{aligned}$$

Thus the coercivity of Φ' implies the boundedness of $\{u_F\}_{F \in \Lambda}$ in V. The proof is complete.

With respect to compact properties of $\{\chi_F\}_{F \in \Lambda}$ we formulate the following result.

Lemma 3.4 For $F \in \Lambda$ let a pair $(u_F, \chi_F) \in F \times L^1(\Omega; R^N)$ satisfy (3.4) and (3.6). Then the set $\{\chi_F \in L^1(\Omega; R^N) : (u_F, \chi_F)$ satisfies (3.4) and (3.6) for some $u_F \in F$, $F \in \Lambda\}$ is weakly precompact in $L^1(\Omega; R^N)$.

Proof. According to the Dunford–Pettis theorem it suffices to show that for each $\varepsilon > 0$ a $\delta_\varepsilon > 0$ can be determined such that for any $\omega \subset \Omega$ with $\mathrm{mes}\, \omega < \delta_\varepsilon$,

$$\int_\omega |\chi_F| d\Omega < \varepsilon, \quad F \in \Lambda. \tag{3.7}$$

Fix $r > 0$ and let $\eta \in R^N$ be such that $|\eta| \le r$. Then we have

$$\chi_F \cdot (\eta - u_F) \le j^0(u_F, \eta - u_F)$$

from which, by virtue of $(H2)$ it results that

$$\chi_F \cdot \eta \le \chi_F \cdot u_F + \alpha(r)(1 + |u_F|^\sigma). \tag{3.8}$$

Denoting by χ_{F_i}, $i = 1, 2, \ldots, N$, the components of χ_F we set

$$\eta = \frac{r}{\sqrt{N}}(\mathrm{sgn}\chi_{F_1}, \ldots, \mathrm{sgn}\chi_{F_N}),$$

where

$$\mathrm{sgn} y = \begin{cases} 1 & \text{if } \ y > 0 \\ 0 & \text{if } \ y = 0 \\ -1 & \text{if } \ y < 0. \end{cases}$$

It is not difficult to verify that $|\eta| \le r$ and

$$\chi_F \cdot \eta \ge \frac{r}{\sqrt{N}}|\chi_F|.$$

Therefore we are led to the estimate

$$\frac{r}{\sqrt{N}}|\chi_F| \le \chi_F \cdot u_F + \alpha(r)(1 + |u_F|^\sigma).$$

Integrating this inequality over $\omega \subset \Omega$ yields

$$\begin{aligned} \int_\omega |\chi_F| d\Omega \le \tfrac{\sqrt{N}}{r} \int_\omega \chi_F \cdot u_F d\Omega + \tfrac{\sqrt{N}}{r}\alpha(r)\mathrm{mes}\,\omega \\ + \tfrac{\sqrt{N}}{r}\alpha(r)(\mathrm{mes}\,\omega)^{\bar{p}} \|u_F\|^\sigma_{L^p(\Omega;R^N)}, \end{aligned} \tag{3.9}$$

where $\bar{p} = \frac{p}{p-\sigma}$. Thus, from (3.9) we obtain

$$\begin{aligned} \int_\omega |\chi_F| d\Omega \ &\le \ \tfrac{\sqrt{N}}{r} \int_\omega \chi_F \cdot u_F d\Omega + \tfrac{\sqrt{N}}{r}\alpha(r)\mathrm{mes}\,\omega + \tfrac{\sqrt{N}}{r}\alpha(r)(\mathrm{mes}\,\omega)^{\bar{p}}\nu^\sigma \|u_F\|^\sigma_V \\ &\le \ \tfrac{\sqrt{N}}{r} \int_\omega \chi_F \cdot u_F d\Omega + \tfrac{\sqrt{N}}{r}\alpha(r)\mathrm{mes}\,\Omega + \tfrac{\sqrt{N}}{r}\alpha(r)(\mathrm{mes}\,\omega)^{\bar{p}}\nu^\sigma M^\sigma, \quad \nu > 0, \end{aligned} \tag{3.10}$$

where ν is a positive constant resulting from the continuous imbedding of $L^p(\Omega; R^N)$ into V. Now we show that

$$\int_\omega \chi_F \cdot u_F d\Omega \le C \tag{3.11}$$

for some positive constant C not depending on $\omega \subset \Omega$ and $F \in \Lambda$. Indeed, from $(H1)$ one can easily deduce that

$$\chi_F \cdot u_F + k(1 + |u_F|) \ge 0 \quad \text{for a.e. } x \in \Omega.$$

Therfore we can write that

$$\int_\omega [\chi_F \cdot u_F + k(1+|u_F|)]d\Omega \le \int_\Omega [\chi_F \cdot u_F + k(1+|u_F|)]d\Omega,$$

and consequently

$$\begin{aligned}\int_\omega \chi_F \cdot u_F d\Omega \ &\le \int_\Omega \chi_F \cdot u_F d\Omega + k \operatorname{mes}\Omega + k\,\nu ||u_F||_V \\ &\le \int_\Omega \chi_F \cdot u_F d\Omega + k \operatorname{mes}\Omega + k\,\nu\, M.\end{aligned}$$

Recall that Φ' maps bounded sets into bounded sets. Accordingly, by means of (3.5) and (3.6) we conclude that

$$\int_\Omega \chi_F \cdot u_F d\Omega = -\langle \Phi'(u_F) - g, u_F\rangle_V \le ||\Phi'(u_F) - g||_{V^*} ||u_F||_V \le \hat{C}, \quad \hat{C} = \text{ const.}$$

From the last two estimates we easily obtain (3.11), as desired. Now let us turn to (3.10). By (3.11) for $r > 0$ one gets

$$\int_\omega |\chi_F| d\Omega \le \frac{\sqrt{N}}{r} C + \frac{\sqrt{N}}{r}\alpha(r)\operatorname{mes}\omega + \frac{\sqrt{N}}{r}\alpha(r)(\operatorname{mes}\omega)^{\bar{p}}\nu^\sigma M^\sigma. \tag{3.12}$$

Now, let $\varepsilon > 0$. Fix $r > 0$ with

$$\frac{\sqrt{N}}{r} C < \frac{\varepsilon}{2}. \tag{3.13}$$

It is a simple matter to determine a $\delta_\varepsilon > 0$ small enough such that

$$\frac{\sqrt{N}}{r}\alpha(r)\operatorname{mes}\omega + \frac{\sqrt{N}}{r}\alpha(r)(\operatorname{mes}\omega)^{\bar{p}}\nu^\sigma M^\sigma \le \frac{\varepsilon}{2} \tag{3.14}$$

whenever $\operatorname{mes}\omega < \delta_\varepsilon$. Finally, from (3.12) $\div$ (3.14) we obtain

$$\int_\omega |\chi_F| d\Omega \le \varepsilon \quad F \in \Lambda,$$

for any $\omega \subset \Omega$ with $\operatorname{mes}\omega < \delta_\varepsilon$. Accordingly, the weak precompactness of $\{\chi_F : F \in \Lambda\}$ in $L^1(\Omega; R^N)$ is proved.

We can now formulate our main result.

Theorem 3.5 Let $(H1) \div (H3)$ be satisfied. Then there exists at least one global minimizer $u \in V$ for I on V such that for some $\chi \in L^1(\Omega; R^N)$ the following holds

$$\langle \Phi'(u) - g, v - u \rangle + \int_\Omega \chi \cdot (v - u) d\Omega = 0 \quad \forall v \in V \cap L^\infty(\Omega; R^N) \qquad (3.15a)$$

$$\chi \in \partial j(u), \quad \chi \cdot u \in L^1(\Omega), \quad j(u) \in L^1(\Omega), \qquad (3.15b)$$

i.e. u is a solution of both (P) and (Q).

Proof. The proof is devided into a sequence of steps. We first show that there exists an element $u \in V$ satisfying (2.7). Next we prove the existence of $\chi \in L^1(\Omega; R^N)$ such that (3.15a) and (3.15b) hold. Finally we invoke Proposition 2.2 to conclude the assertion.

Step 1. For $F \in \Lambda$ let

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} \{(u_{F'}, \chi_{F'}) \in V \times L^1(\Omega; R^N) : (u_{F'}, \chi_{F'}) \text{ satisfies } (3.4) and (3.6)\}.$$

(with F being replaced by F' in (3.4) and (3.6)). We use the symbol weakcl(W_F) to denote the closure of W_F in the weak topology of $V \times L^1(\Omega; R^N)$. Moreover, let

$$Z = \bigcup_{F \in \Lambda} \{\chi_F \in L^1(\Omega; R^N) : (u_F, \chi_F) \text{ satisfies } (P_F) \text{ for some } u_F \in F\}.$$

Denoting by weakcl (Z) the closure of Z in the weak topology of $L^1(\Omega; R^N)$ we get

$$\text{weakcl}\,(W_F) \subset B_V(O, M) \times \text{weakcl}\,(Z) \quad \forall F \in \Lambda.$$

Since $B_V(O, M)$ is weakly compact in V and, by Lemma 3.4, weakcl (Z) is weakly compact in $L^1(\Omega; R^N)$, the family $\{\text{weakcl}(W_F) : F \in \Lambda\}$ is contained in the weakly compact set $B_V(O, M) \times \text{weakcl}(Z)$ in $V \times L^1(\Omega; R^N)$. Now let us notice that for any $F_1, \ldots, F_k \in \Lambda$, $k = 1, 2, \ldots$, we have the inclusion $W_{F_1} \cap \ldots \cap W_{F_k} \supset W_F$, with $F = F_1 + \ldots + F_k$, from which it follows by Proposition 3.3 that the family $\{\text{weakcl}(W_F) : F \in \Lambda\}$ has the finite intersection property. Thus the intersection

$$\bigcap_{F \in \Lambda} \text{weakcl}\,(W_F)$$

is not empty. Let (u, χ) be an element of this intersection. It is clear by the procedure employed that u is a solution of the optimization problem (2.7) and that $j(u) \in L^1(\Omega)$. What is left is to show that (u, χ) satisfies (3.15). Let us fix $v \in V \cap L^\infty(\Omega; R^N)$ arbitrarily. We choose $F \in \Lambda$ such that $v \in F$. Thus there exists a sequence $\{(u_{F_n}, \chi_{F_n})\}$ in $F_n \times L^1(\Omega; R^N) \subset W_F$ (for simplicity of notations it will be denoted by (u_n, χ_n)) with the properties that

$$\begin{gathered}\langle \Phi'(u_n) - g, w - u_n \rangle + \int_\Omega \chi_n \cdot (w - u_n) d\Omega = 0 \quad \forall w \in F_n \\ \chi_n \in \partial j(u_n)\end{gathered} \tag{3.16}$$

and

$$\begin{aligned} u_n &\to u \quad \text{weakly in } V \\ \chi_n &\to \chi \quad \text{weakly in } L^1(\Omega; R^N). \end{aligned} \tag{3.17}$$

Taking into account (3.16) and the fact that $v \in F \subset F_n$ we have

$$\langle \Phi'(u_n) - g, v \rangle + \int_\Omega \chi_n \cdot v d\Omega = 0, \quad n = 1, 2 \dots . \tag{3.18}$$

Now we are in a position to pass to the limit as $n \to \infty$ in (3.18). The boundedness of $\Phi'(u_n)$ allows us to conclude that for some $B \in V^\star$, $\Phi'(u_n) \to B$ weakly in $V^\star$ (by passing to a subsequence, if necessary). Since $v \in V \cap L^\infty(\Omega; R^N)$ has been chosen arbitrarily, the equality

$$\langle B - g, v \rangle + \int_\Omega \chi \cdot v d\Omega = 0 \tag{3.19}$$

is valid for any $v \in V \cap L^\infty(\Omega; R^N)$.

Step 2. We prove that the first claim in (3.15*b*) holds. Since V is compactly imbedded into $L^p(\Omega; R^N)$, from (3.17) we obtain (by passing to a subsequence, if necessary)

$$u_n \to u \quad \text{strongly in } L^p(\Omega; R^N). \tag{3.20}$$

This implies that for a subsequence of $\{u_n\}$ (again denoted by the same symbol) one gets

$$u_n \to u \quad \text{a.e. in } \Omega.$$

Thus Egoroff's theorem can be applied from which it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $\operatorname{mes} \omega < \varepsilon$ can be determined such that

$$u_n \to u \quad \text{uniformly in } \Omega \setminus \omega$$

with $u \in L^\infty(\Omega \setminus \omega; R^N)$. Let $v \in L^\infty(\Omega \setminus \omega; R^N)$ be an arbitrary function. From the estimate

$$\int_{\Omega \setminus \omega} \chi_n \cdot v d\Omega \leq \int_{\Omega \setminus \omega} j^0(u_n, v) d\Omega$$

combined with the weak convergence in $L^1(\Omega; R^N)$ of χ_n to χ , (3.20) and with the upper semicontinuity of

$$L^\infty(\Omega \setminus \omega; R^N) \ni w \longmapsto \int_{\Omega\setminus\omega} j^0(w,v)d\Omega$$

we obtain

$$\int_{\Omega\setminus\omega} \chi \cdot v d\Omega \leq \int_{\Omega\setminus\omega} j^0(u,v)d\Omega \quad \forall v \in L^\infty(\Omega \setminus \omega; R^N).$$

But the last inequality amounts to

$$\chi \in \partial j(u) \quad \text{a.e. in } \Omega \setminus \omega.$$

Since $\operatorname{mes}\omega < \varepsilon$ and ε was chosen arbitrarily,

$$\chi \in \partial j(u) \quad \text{a.e. in } \Omega, \tag{3.21}$$

as claimed.

Step 3. Now we show that $\chi \cdot u \in L^1(\Omega)$. According to the Truncation Hypothesis one can find sequences $\{\varepsilon_k\} \in L^\infty(\Omega)$ and $\{\kappa_k\} \in L^\infty(\Omega; R^N)$ with $0 \leq \varepsilon_k \leq 1$, $||\kappa_k||_{L^\infty(\Omega;R^N)} \leq C$, such that $\hat{u}_k := (1-\varepsilon_k)u + \varepsilon_k\kappa_k \in V \cap L^\infty(\Omega; R^N)$ and $\hat{u}_k \to u$ in V. Since we already know that $\chi \in \partial j(u)$, one can apply $(H1)$ to obtain

$$\chi \cdot (-u) \leq j^0(u; -u) \leq k(1 + |u|).$$

Hence

$$\chi \cdot u \geq -k(1 + |u|) \tag{3.22}$$

and consequently,

$$\chi \cdot \hat{u}_k = \chi \cdot ((1 - \varepsilon_k)u + \varepsilon_k\kappa_k) \geq -k(1 + |u|) - C|\chi|. \tag{3.23}$$

This implies that the sequence $\{\chi \cdot \hat{u}_k\}$ is bounded from below by a function which is integrable in Ω. On the other hand, due to (3.19) we get

$$\hat{C} \geq \langle -B + g, \hat{u}_k \rangle = \int_\Omega \chi \cdot \hat{u}_k d\Omega.$$

Thus by Fatou's lemma $\chi \cdot u \in L^1(\Omega)$, as required.

Step 4. In this step we are aimed at establishing the esimate

$$\liminf \int_\Omega \chi_n \cdot u_n d\Omega \geq \int_\Omega \chi \cdot u d\Omega. \tag{3.24}$$

For this purpose let us fix $v \in L^\infty(\Omega; R^N)$ arbitrarily. Since $\chi_n \in \partial j(u_n)$, one gets by $(H2)$

$$\chi_n \cdot (v - u_n) \le j^0(u_n; v - u_n) \le \alpha(||v||_{L^\infty(\Omega;R^N)})(1 + |u_n|^\sigma).$$

Thus by Fatou's lemma and by the semicontinuouity of $V \ni w \mapsto \int_\Omega j^0(u; w-u)d\Omega$, we obtain

$$\liminf \int_\Omega \chi_n \cdot u_n d\Omega \ge \int_\Omega \chi \cdot v d\Omega - \int_\Omega j^0(u; v-u)d\Omega \quad \forall v \in V \cap L^\infty(\Omega; R^N). \tag{3.25}$$

On substituting $v = \hat{u}_k$ into the right hand side of (3.25) one gets

$$\liminf_n \int_\Omega \chi_n \cdot u_n d\Omega \ge \liminf_k \int_\Omega \chi \cdot \hat{u}_k d\Omega - \limsup_k \int_\Omega j^0(u; \hat{u}_k - u)d\Omega. \tag{3.26}$$

Further, it is easy to check that

$$j^0(u; \hat{u}_k - u) = \varepsilon_k j^0(u; \kappa_k - u) \le \varepsilon_k \alpha(||\kappa_k||_{L^\infty(\Omega;R^N)})(1 + |u|^\sigma) \le \alpha(C)(1 + |u|^\sigma). \tag{3.27}$$

Therefore due to the fact that $\hat{u}_k \to u$ a.e. in Ω we are allowed to apply Fatou's lemma to deduce

$$\limsup_k \int_\Omega j^0(u; \hat{u}_k - u)d\Omega \le 0,$$

whereas from (3.23) we get

$$\liminf_k \int_\Omega \chi \cdot \hat{u}_k d\Omega \ge \int_\Omega \chi \cdot u d\Omega.$$

Finally, combining these inequalities with (3.26) yields (3.24), as desired.

Step 5. We proceed to establish (3.15a). Our first claim is that

$$\langle B - g, u\rangle + \int_\Omega \chi \cdot u d\Omega = 0. \tag{3.28}$$

Indeed, (3.19) yields

$$\langle B - g, \hat{u}_k\rangle + \int_\Omega \chi \cdot \hat{u}_k d\Omega = 0.$$

Since $\chi \cdot u \in L^1(\Omega)$ we get the following estimate

$$\chi \cdot \hat{u}_k = (1 - \varepsilon_k)\chi \cdot u + \varepsilon_k \chi \cdot \kappa_k \le |\chi \cdot u| + C|\chi|$$

which together with (3.23) permits the conclusion

$$\int_{\Omega} \chi \cdot \hat{u}_k d\Omega \to \int_{\Omega} \chi \cdot u d\Omega$$

due to dominated convergence. Thus the assertion follows. Next the pseudo-monotone property of Φ' will be used. One can check at once that from (3.16) and (3.28) it follows that

$$\langle \Phi'(u_n), u_n - u\rangle \leq \langle B - \Phi'(u_n), u\rangle + \langle g, u_n - u\rangle + \int_{\Omega} \chi \cdot u d\Omega - \int_{\Omega} \chi_n \cdot u_n d\Omega.$$

Hence

$$\limsup \langle \Phi'(u_n), u_n - u\rangle \leq -\liminf \int_{\Omega} \chi_n \cdot u_n d\Omega + \int_{\Omega} \chi \cdot u d\Omega \leq 0$$

by means of (3.24). The known properties of pseudo-monotone mappings amount to $B = \Phi'(u)$ and $\langle \Phi'(u_n), u_n\rangle \to \langle \Phi'(u), u\rangle$. Accordingly, to complete the proof it suffices to apply the limit procedure in (3.16) and to invoke Proposition 2.2.

4 Final Remarks and Comments

In the present approach the function space V was assumed to be vector–valued. Essential simplifications are possible if $N = 1$ and V is compactly imbedded into $L^p(\Omega)$, $p > 1$. Namely, $(H2)$ turns out to be superfluous because $(H1)$ implies $(H2)$ with $\sigma = 1$ [15, 18]. Further, the famous result of Hedberg [10] ensures that each Sobolev space $V = W^{l,p}(\Omega)$ satisfies $(H3)$ with $\kappa_k = 0$ provided the boundary $\partial\Omega$ is smooth enough. Accordingly we are led to the result being a consequence of the foregoing remarks and the Sobolev imbedding theorem.

Theorem 4.1 Let $\Omega \subset R^m$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$. Assume that $V = W^{l,q}(\Omega), q > 1, l \geq 1$, and that $(H1)$ holds. Then I attains its infimum over V at some $u \in V$. Moreover, there exists $\chi \in L^1(\Omega)$ such that the pair (u, χ) satisfies $(3.15a)$ and $(3.15b)$.

For other applications of the Hedberg truncation technique to study nonlinear problems the reader is referred to [3, 4, 29] and the references quoted there.

Let us turn to the vectorial case $N > 1$. According to the author's knowledge the truncation conjecture $(H3)$ for a Sobolev vector space $V = W^{l,q}(\Omega; R^N)$, $N > 1$, seems to be an open problem.

The unilateral growth restrictions $(H1)$ and $(H2)$ have been introduced by the author in [15] (see also [16, 18]) in order to generalize the well known sign condition (see for instance [4, 29]) to the case in which nonsmoothness occures and the original space consists of vector–valued functions.

It is worth noting that, up till now, all the existence results for hemivariational inequalities involving unilateral growth hypotheses of the type $(H1)$ and $(H2)$ have been derived on the condition that $\Phi'(\cdot)$ is weakly continuous, i.e. it maps weakly converging sequences of V into weakly converging sequences of $V^\star$. In the present approach we have strengthened the result because $\Phi'(\cdot)$ was assumed to be pseudo–monotone.

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